Oct 7, 2022

Week 5

2020A Adv. Cal. II

Triple integral does not differ much from double integral. We'll be brief.

Let

$$B = [a,b]\times[c,d]\times[e,f] \subset \mathbb{R}^3$$

be a rectangular box. A partition P on B is

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_m = d$$
$$e = z_0 < z_1 < \cdots < z_\ell = f.$$

Let $B_{ijk} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]\times[z_{k-1}, z_k]$ be sub-rectgle box. For a fcn $f = f(x,y,z)$ in B, its Riemann sum is

$$S(f,P) = \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

f is called integrable in B if there exists a number I satisfying whenever $\|P\| \to 0$,

$$S(f,P) \to I$$

for all tags $(x_i^*, y_j^*, z_k^*) \in B_{ijk}$. In math. formulation, ie, for given $\varepsilon > 0$, $\exists\, \delta$ s.t.

$$|S(f,P) - I| < \varepsilon, \quad \forall P, \ \|P\| < \delta.$$

Here $\|P\| = \max\{\Delta x_i, \Delta y_j, \Delta z_k\}$. Denote $I = \iiint_B f$.

- Integrable fcns are bounded.
- There are bounded non-integrable fcns
- All continuous / piecewise continuous fcns are integrable.

Fubini's thm Let f be piecewise continuous in B. Then

$$\iiint_B f = \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y).$$

$$= \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx, \quad R=[a,b]\times[c,d].$$

Idea of pf:

$$\iiint_B f \sim \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

$$= \sum_{i,j} \Big[\sum_k f(x_i^*, y_j^*, z_k^*)\,\Delta z_k\Big]\,\Delta x_i \Delta y_j$$

$$\sim \sum_{i,j} \int_e^f f(x_i^*, y_j^*, z)\,dz\,\Delta x_i \Delta y_j$$

$$\sim \iint_R \Big(\int_e^f f(x,y,z)\,dz\Big)\,dA(x,y).$$

Let Ω be a region in $\mathbb{R}^3$ and f a fcn defined on it. Let

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z)\in\Omega \\ 0, & (x,y,z)\notin\Omega \end{cases}$$

be the universal extension of f.

Define

$$\iiint_{\Omega} f = \iiint_{B} \tilde{f}, \quad \Omega \subset B.$$

The definition makes sense as long as f is piecewise continuous in Ω.

<u>Theorem</u> Let Ω be described as

$$g_1(x,y) \le z \le g_2(x,y)$$
$$(x,y) \in D.$$

For piecewise continuous f in Ω,

$$\iiint_{\Omega} f = \iint_{D} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dA(x,y).$$

When $D \subset \mathbb{R}^2$ is described as

$$g_1(x) \le y \le g_2(x)$$
$$a \le x \le b$$

then

$$\iint_{D} f = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

This formula is the exact analog of the 2-D formula.

e.g. Describe the ball

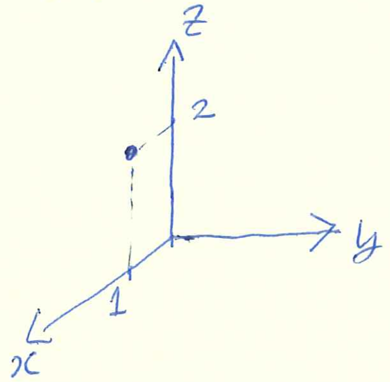
$$(x-1)^2 + y^2 + (z-2)^2 \le 9$$

as a region in this theorem.

The boundary of this ball is the sphere

$$(x-1)^2+y^2+(z-2)^2=9,$$

whose center is $(1,0,2)$ and radius 3.



Its projection onto the xy-plane is $(x-1)^2+y^2=9$, so D is the disk at $(1,0)$, with radius 3.

Here $(z-2)^2 = 9-(x-1)^2-y^2$

$$z = 2 \pm \sqrt{9-(x-1)^2-y^2}$$

Take $g_1(x,y) = 2-\sqrt{9-(x-1)^2-y^2}$, $g_2(x,y) = 2+\sqrt{9-(x-1)^2-y^2}$

$\therefore$ Ω is described as:

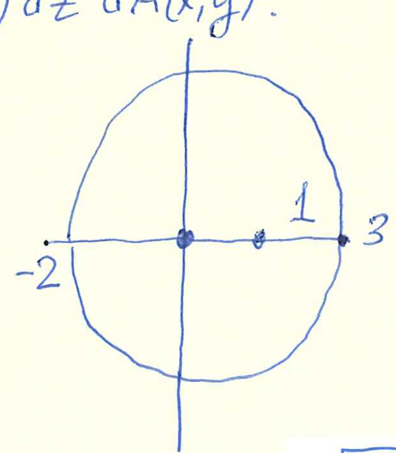
$$2-\sqrt{9-(x-1)^2-y^2} \leq z \leq 2+\sqrt{9-(x-1)^2-y^2},$$

$$D: (x-1)^2+y^2 \leq 9$$

In general,

$$\iiint_\Omega f = \iint_D \int_{2-\sqrt{9-(x-1)^2-y^2}}^{2+\sqrt{9-(x-1)^2-y^2}} f(x,y,z)\,dz\,dA(x,y).$$

$$= \int_{-2}^{3}\int_{-\sqrt{9-(x-1)^2}}^{\sqrt{9-(x-1)^2}}\int_{2-\sqrt{9-(x-1)^2-y^2}}^{2+\sqrt{9-(x-1)^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$



$y = \pm\sqrt{9-(x-1)^2}$

e.g. Let Ω be the solid bdd by $z=x^2+y^2$ and $z=x$. Describe Ω by determing g_1, g_2 and D.

The solid is over D whose boundary satisfies

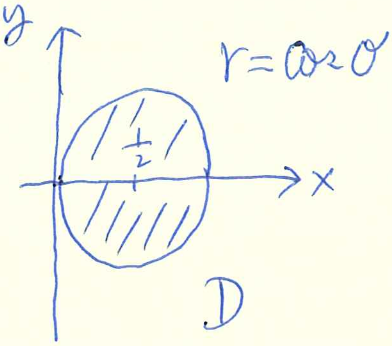
$$x^2+y^2=x, \text{ ie}$$

$$(x-\tfrac{1}{2})^2+y^2=\tfrac{1}{4}.$$

$\therefore D=\{(x,y):(x-\frac{1}{2})^2+y^2\le\frac{1}{4}\}$ a disk, since $x^2+y^2\le x$ over D, $g_1(x,y)=x^2+y^2$ and $g_2(x,y)=x$.

$$\iiint_\Omega f=\iint_D\int_{x^2+y^2}^{x} f(x,y,z)\,dz\,dA(x,y).$$

For instance, let $f\equiv1$, the volume of Ω is



$$\iint_D\int_{x^2+y^2}^{x} 1\,dz\,dA(x,y)$$

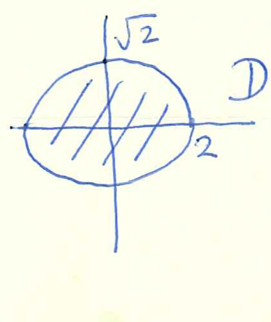
$$=\iint_D (x-x^2-y^2)\,dA(x,y)$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(r\cos\theta-r^2)\,r\,dr\,d\theta$$

$$=\cdots \#$$

eg. Let Ω be the solid bounded between $z=8-x^2-y^2$, $z=x^2+3y^2$. Find its volume.

This two surfaces intersect at



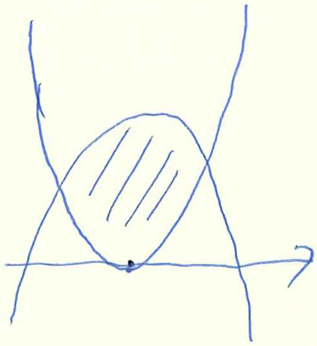
$$8-x^2-y^2=x^2+3y^2, \text{ ie}$$

$$x^2+2y^2=4.$$

At $(x,y)=(0,0)$ $\quad 8-x^2-y^2 = 8 > x^2+3y^2 = 0$, so

$$g_1(x,y) = x^2+3y^2, \quad g_2(x,y) = 8-x^2-y^2.$$

$$\text{Vol.} = \iint_D \int_{x^2+3y^2}^{8-x^2-y^2} 1 \, dz \, dA(x,y)$$



a cross section of Ω

$$= \iint_D (8-2x^2-4y^2) \, dA(x,y)$$

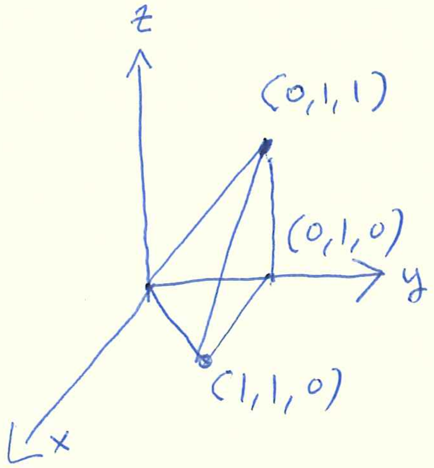
$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2) \, dy \, dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{2} (4-x^2)^{3/2} \, dx$$

$$\vdots$$

$$= 8\pi\sqrt{2} \quad \#$$

e.g. Let T be the tetrahedron with vertices at $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$



equations for 4 faces are

$$y = x+z, \quad y=1, \quad x=0, \quad z=0.$$

(identify them one by one.)

Express

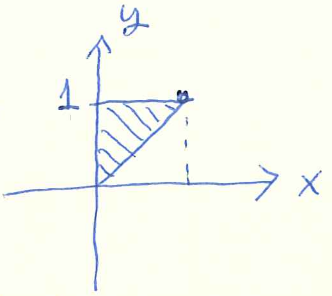
$$\iiint_T f$$

in the order of $dz\,dx\,dy$ and $dx\,dy\,dz$.

In the $dz\,dx\,dy$ case we regard T as solid over xy-plane.

$T: 0 \leq z \leq -x+y$

$(x,y) \in$ the face D :



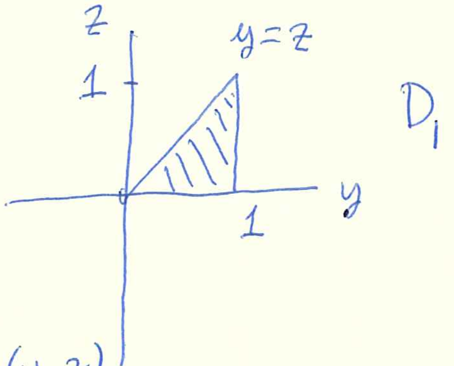
$$\therefore \iiint_T f = \iint_D \int_0^{-x+y} f(x,y,z)\,dz\,dA(x,y)$$

$$= \int_0^1 \int_0^y \int_0^{-x+y} f(x,y,z)\,dz\,dx\,dy.$$

Next, the $dx\,dy\,dz$ case is to project T onto yz-plane

$T: 0 \leq x \leq y-z$

D_1 is in the yz-plane



$$\therefore \iiint_T f = \iint_{D_1} \int_0^{y-z} f(x,y,z)\,dx\,dA(y,z)$$

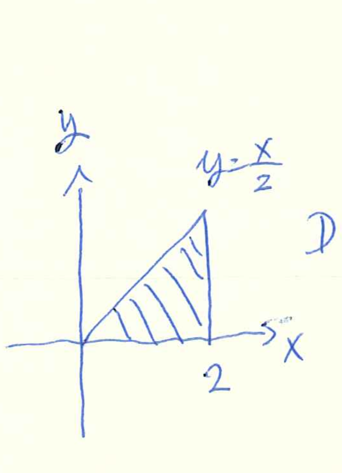
$$= \int_0^1 \int_z^1 \int_0^{y-z} f(x,y,z)\,dx\,dy\,dz. \#$$

e.g. Evaluate

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{\cos x^2}{2\sqrt{z}}\,dx\,dy\,dz.$$

$\int \cos^2 x\,dx$ difficut, so we switch

$$\int_0^1 \int_{2y}^2 \frac{\cos x^2}{2\sqrt{z}}\,dx\,dy$$



$$= \iint_D \frac{\cos x^2}{2\sqrt{z}}\, dA(x,y)$$

$$= \int_0^2 \int_0^{x/2} \frac{\cos x^2}{2\sqrt{z}}\, dy\, dx$$

$$= \int_0^2 \frac{1}{2\sqrt{z}} \frac{x}{2} \cos x^2\, dx$$

$$= \frac{1}{2\sqrt{z}} \frac{1}{4} \sin 4$$

$\therefore$ our integral

$$= \int_0^4 \frac{1}{2\sqrt{z}} \frac{1}{4} \sin 4\, dz$$

$$= \frac{\sin 4}{8} \int_0^4 \frac{1}{\sqrt{z}}\, dz$$

$$= \frac{1}{2} \sin 4 \quad \#$$

(Cont'd)

Moments and Center of Mass

It suffices to memorise some definition.

Mass $M = \iiint_\Omega \delta\, dV$, δ – density fcn, Ω – solid

First moments about the coordinates planes

$$M_{yz} = \iiint_\Omega x\,\delta\, dV, \quad M_{xz} = \iiint_\Omega y\,\delta\, dV, \quad M_{xy} = \iiint_\Omega z\,\delta\, dV.$$

Center of mass $\vec{c} = (\bar{x}, \bar{y}, \bar{z})$,

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

When $\delta = \text{const.}$, the center of mass is the centroid.

2-dim analog:

$$M_y = \iint_D x\,\delta\,dA, \quad M_x = \iint_D y\,\delta\,dA.$$

$$\vec{c} = (\bar{x}, \bar{y}), \quad \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

moments of inertia
(second moments)

about the x-axis $\quad I_x = \iiint_\Omega (y^2+z^2)\,\delta\,dV$

about the y-axis $\quad I_y = \iiint_\Omega (x^2+z^2)\,\delta\,dV$

about the z-axis $\quad I_z = \iiint_\Omega (x^2+y^2)\,\delta\,dV$

2-dim analog

about the x-axis $\quad I_x = \iint_D y^2\,\delta\,dA$

about the y-axis $\quad I_y = \iint x^2\,\delta\,dA$

about the origin $\quad I_0 = I_x + I_y.$

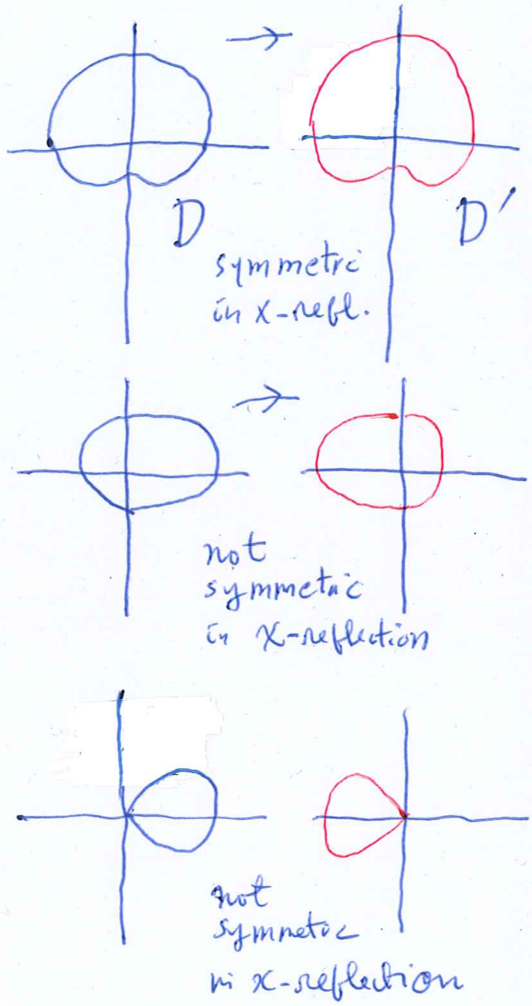
Consider the x-reflection (reflection with respect to the x-axis:

$$(x,y) \longmapsto (-x,y)$$

a region D goes to another region D'

$$D' = \{(x,y) : (-x,y) \in D\}.$$

In case $D' = D$, D is symmetric w.r.t. x-reflection.



<u>Theorem</u> Let f be in an x-symmetric region D satisfying

$$f(-x,y) = -f(x,y), \quad (x,y) \in D$$

(odd in x). Then

$$\iint_D f(x,y)\, dA(x,y) = 0.$$

Similarly, if f is odd in x,

$$f(-x,y,z) = -f(x,y,z), \quad (x,y) \in \Omega$$

when Ω is x-symmetric, then

$$\iiint_\Omega f(x,y,z)\, dV(x,y,z) = 0.$$

▨ Pf: We consider 3-dim case. Pick a large box

$$B = [-a,a] \times [-c,c] \times [-d,d],$$

to contain Ω. The universal extension of f

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in \Omega, \\ 0, & (x,y,z) \notin \Omega. \end{cases}$$

still satisfies

$$\tilde{f}(-x,y,z) = -\tilde{f}(x,y,z).$$

So

$$\iiint_\Omega f\,dV \overset{\text{df}}{=} \iiint_B \tilde{f}(x,y,z)\,dV$$

$$= \int_{-c}^{c}\int_{-e}^{e}\int_{-a}^{a} \tilde{f}(x,y,z)\,dx\,dz\,dy.$$

$$\int_{-a}^{a} \tilde{f}(x,y,z)\,dx = \int_{-a}^{0} \tilde{f}(x,y,z)\,dx + \int_{0}^{a} \tilde{f}(x,y,z)\,dx$$

$$= \int_{a}^{0} \tilde{f}(-s,y,z)(-ds) + \int_{0}^{a} \tilde{f}(x,y,z)\,dx$$

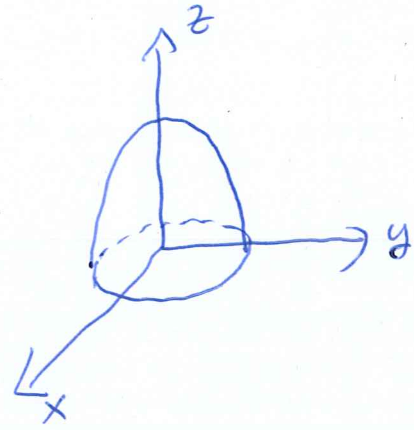
$$= \int_{0}^{a} \tilde{f}(-s,y,z)\,ds + \int_{0}^{a} \tilde{f}(x,y,z)\,dx$$

$$= -\int_{0}^{a} \tilde{f}(s,y,z)\,ds + \int_{0}^{a} \tilde{f}(x,y,z)\,dx$$

$$= 0.$$

$$\therefore \iiint_\Omega f\,dV = \int_{-c}^{c}\int_{-e}^{e}\int_{-a}^{a} \tilde{f}(x,y,z)\,dx\,dz\,dy = 0 \quad . \blacksquare$$

e.g. Find the centroid of Ω which is bounded by $z = 4 - x^2 - y^2$ over the xy-plane.



Ω is described by

$$0 \leq z \leq 4 - x^2 - y^2$$

$$x^2 + y^2 \leq 4.$$

$$M = \iiint_\Omega 1 \, dV = \iint_{D_2} \int_0^{4-x^2-y^2} 1 \, dz \, dA(x,y)$$

$$= \iint_{D_2} (4 - x^2 - y^2) \, dA(x,y)$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) \, r \, dr \, d\theta$$

$$= 8\pi.$$

Next,

$$M_{xy} = \iiint_\Omega z \, dV = \iint_{D_2} \int_0^{4-x^2-y^2} z \, dz \, dA(x,y)$$

$$= \frac{1}{2} \iint_{D_2} (4 - x^2 - y^2)^2 \, dA(x,y)$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 \, r \, dr \, d\theta$$

$$\vdots$$

$$= \frac{32\pi}{3}.$$

On the other hand, D_2 is x-symmetric and y-symmetric. Moreover, the function $f(x,y,z)=x$ satisfies $f(-x,y,z)=-f(x,y,t)$. By the theorem above,

$$M_{yz}=\iiint_{\Omega} x\,dV=0.$$

Similarly, $f(x,-y,z)=-f(x,y,z)$ when $f(x,y,z)=y$,

$$\therefore M_{xz}=\iiint_{\Omega} y\,dV=0.$$

$$\therefore (\bar{x},\bar{y},\bar{z})=\left(0,0,\frac{32\pi}{3}\Big/8\right)$$

$$=(0,0,4/3).\ \#$$